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LIMITING STATIONARY MODES OF A RIGID BODY ON A STRING SUSPENSION*

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The nature of the stationary rotations of an axisymmetric rigid body on a string suspension at high angular velocities is studied. The domain in the space of dimensionless parameters of the system is found where, given the same angular velocity, the maximum number (sixteen) of different permanent rotations is possible.

It was shown earlier, see /1/** (**and also ISHLINSKII A. YU., et al., On a method of balancing a rapidly rotating body, Preprint No.146, Inst. Problem Mekhaniki Akad. Nauk SSSR, Moscow, 1980), that as the angular velocity increases without limit, the system tends to a position in which some principal central axis of inertia of the body coincides with the fixed vertical. It was shown in /2/ that there are stationary motions in which an axis of symmetry of the body is horizontal. The different types of permanent rotations have been studied e.g., in /2, 3/*** (***see also SARYCHEV V.A., et al., The positions of relative equilibrium of an axisymmetric rigid body suspended on a string, Preprint No.140, Inst. Prikl. Matem. Akad. Nauk SSSR, Moscow, 1987).

We consider an axisymmetric rigid body with centre of mass G, suspended on a weightless absolutely rigid rod at an arbitrary point O_2 (Fig.1). The other end of the rod (point O_1) is coupled to a device that rotates the system with angular velocity ω about a vertical axis. The equations describing the permanent rotations of the system are /2/

$$\omega^2 (l \sin\alpha + a\sin\theta)\cos\alpha - g\sin\alpha = 0$$
$$m\omega^2 a (l\sin\alpha + a\sin\theta)\cos\theta + {}^1/_2 (A - C)\,\omega^2 \sin 2(\delta + \theta) = mga\sin\theta \tag{1}$$

where m is the mass of the body, A, C are the central equatorial and axial moments of inertia of the body, a is the distance O_2G, and l is the length of the rod O_1O_2. Let α, θ be respectively the angles between the downwards vertical and the vectors O_1O_2 and O_2G. The angle δ between O_2G and the axis of symmetry is best regarded as always positive and varying in the interval $[0, \pi/2]$, regardless of whether the point O_2 is above or below the axis of symmetry. In order to distinguish between these cases, we make the following stipulation. We measure δ counter-clockwise from the line O_2G. If δ is then greater than $\pi/2$, we consider the position of the system after a half-period. Then, δ will lie in the required range, and α and θ will be replaced by $-\alpha, -\theta$. This means that both α and θ must be regarded as varying in the interval $(-\pi, \pi]$.

We introduce the parameters

$$z_1 = \frac{\omega^2 a}{g} + \frac{(A-C)\,\omega^2}{mga}\cos 2\delta, \quad z_3 = \frac{(A-C)\,\omega^2}{mga}\cos 2\delta$$
$$z_2 = \frac{(A-C)\,\omega^2}{2mga}\sin 2\delta, \quad z_4 = \frac{\omega^2 l}{g}$$

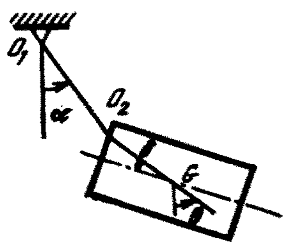

Fig.1

and we write system (1) in the form

$$[(z_1 - z_3)\sin\theta + z_4\sin\alpha]\cos\alpha = \sin\alpha$$
$$z_1\sin\theta\cos\theta + z_4\sin\alpha\cos\theta + z_2\cos 2\theta = \sin\theta \tag{2}$$

On eliminating the angle α from (2) and introducing the variable $u = \operatorname{tg} {}^1/_2\theta$, we obtain the equation

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$$f^2(u; z_1, z_2)\,[f^2(u; z_3, z_2) + (1-u^4)^2] - z_4^2(1-u^4)^2 \times f^2(u; z_3, z_2) = 0 \tag{3}$$
$$f(u; x, y) = 2xu(1-u^2) + y(u^4 - 6u^2 + 1) - 2u(1+u^2)$$

the left-hand side of which is a polynomial of degree sixteen in u. This means that Eq.(3) can have at most 16 solutions, and the mechanical system can have at most 16 stationary motions. It is not yet clear whether a system exists for which all 16 soltuions can be realized. Further analysis not only answers this question in the affirmative, but also indicates the domain of parameter space in which this is possible.

We consider Eq.(3) for large ω. For this, we introduce the small parameter $\varepsilon = 1/\omega^2$, put $z_i^\circ = \varepsilon z_i$ $(i = 1, 2, 3, 4)$, and seek the solution as the series

$$u = u_0 + \varepsilon u_1 + \ldots \tag{4}$$

To find u_0 we substitute series (4) into Eq.(3) and retain only the terms which are independent of ε. The resulting equation will hold when u_0 is a solution of any of the three equations

$$2z_1^\circ u_0(1-u_0^2) + z_2^\circ(u_0^4 - 6u_0^2 + 1) + \sigma z_4^\circ(1-u_0^4) = 0, \quad \sigma = \pm 1 \tag{5}$$

$$2z_3^\circ u_0(1-u_0^2) + z_2^\circ(u_0^4 - 6u_0^2 + 1) = 0 \tag{6}$$

We find from system (2) that, as $\omega \to \infty$, we have $\alpha_0 = \pi/2$ for motions given by Eq.(5) with $\sigma = 1$, while $\alpha_0 = -\pi/2$ when $\sigma = -1$, and

$$\sin\alpha_0 = -\frac{2u_0(z_1^\circ - z_3^\circ)}{(1+u_0^2)z_4^\circ} = -\frac{z_1^\circ - z_3^\circ}{z_4^\circ}\sin\theta_0 \tag{7}$$

when u_0 satisfies Eq.(6).

We start by considering Eq.(6). We write it in terms of the physical parameters, substituting $z_2^\circ/z_3^\circ = {}^1/_2\operatorname{tg} 2\delta$, $u_0 = \operatorname{tg} {}^1/_2\theta_0$. It can be shown that the result is equivalent to the equation

$$\operatorname{tg} 2\theta_0 + \operatorname{tg} 2\delta = 0$$

the four roots of which satisfy the equations

$$1)\ \theta_0 + \delta = \pi, \quad 2)\ \theta_0 + \delta = \pi/2, \quad 3)\ \theta_0 + \delta = 0, \quad 4)\ \theta_0 + \delta = -\pi/2 \tag{8}$$

By (7), we have here for the angle α_0 the equation

$$\sin\alpha_0 = -al^{-1}\sin\theta_0 \tag{9}$$

i.e., in the **zeroth** approximation the centre of mass of the body lies on the axis of rotation of the system.

Expressions (8) and (9) imply that there are types of stationary motions for which, as ω increases without limit, a principal central axis of inertia of the body tends to coincidence with the fixed vertical. This property, which was studied theoretically in /1/, is the basis of the method of dynamic balancing, the aim of which is to find the principal central axis of inertia, and possibly to check that it coincides with the axis of geometrical symmetry.

Returning to Eq.(6), we write its four solutions as

$$u_{0\pm} = (v \pm \sqrt{v^2+4})/2, \quad v_\pm = p_1 \pm \sqrt{p_1^2+4}, \quad p_1 = z_3/z_2 \tag{10}$$

(Here and throughout, we omit the subscript $\circ$ in the parameters $z_1, \ldots, z_4$). Note that, corresponding to Eqs.1)-4) of (8) we have the relations

$$1)\ u_+(v_-) > 1, \quad 2)\ 0 < u_+(v_-) < 1, \quad 3)\ u_-(v_+) = -1/u_+(v_+), \quad 4)\ u_-(v_-) = -1/u_+(v_-) \tag{11}$$

Substituting series (4) into Eq.(3) and retaining terms with ε, we get

$$\frac{1}{(cu_1+b)^2} = \frac{1}{[2p_2u_0(1-u_0^2)]^2} - \frac{1}{(1-u_0^4)^2}, \quad p_2 = \frac{z_1 - z_3}{z_4} \tag{12}$$

where c and b are expressions which depend on u_0, z_i. In the case when

$$(1+u_0^2)^2 - (2p_2u_0)^2 > 0 \tag{13}$$

when Eq.(12) has real solutions u_1, there are eight types of permanent rotations defined in the zeroth approximation by Eqs.(8) or their equivalents (10). The solutions in each pair that corresponds to the same u_0, differ by the value of u_1 and the sign of $\cos\alpha_0$ (Fig.2).

We study (13) by factorizing its left-hand side. Since $p_2 = a/l > 0$, (13) is equivalent to

$$(1 + u_0^2) - \sigma\cdot 2p_2u_0 > 0 \tag{14}$$

where $\sigma = 1$ if $u_0 > 0$, and $\sigma = -1$ if $u_0 < 0$.

Assume that, for $u_0 > 0$ we have (14) with $\sigma = 1$; then obviously, for $-1/u_0$ we have (14) with $\sigma = -1$. Recalling that (11) hold for the values (10), it suffices to require simply that (14) holds for $\sigma = 1$ with positive u_0: $u_+(v_+)$ and $u_+(v_-)$. By solving (14) for $\sigma = 1$ and using (10) for each root, we obtain the domains in the space of parameters p_1, p_2, in which (14) holds. These domains are given by the sets of inequalities

$$\begin{cases} L > 0 \\ 0 < p_2 < \sqrt{2} \end{cases} \text{ or } \begin{cases} L < 0 \\ \sigma p_1 > 0 \end{cases} \tag{15}$$
$$L = (2 - p_2^2)^2 + p_1^2(1 - p_2^2)$$

when $\sigma = 1$ for the root $u_+(v_+)$ and $\sigma = -1$ for $u_+(v_-)$.

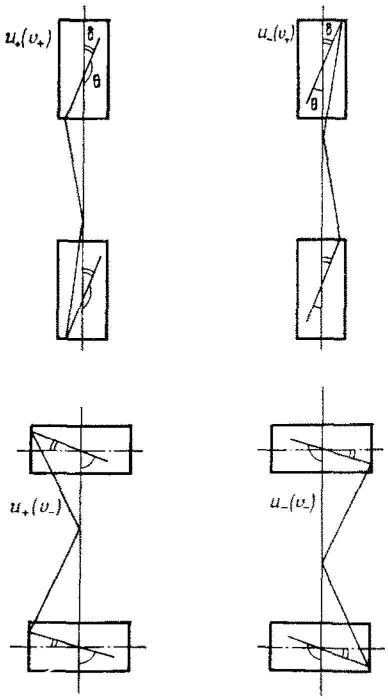

Fig.2

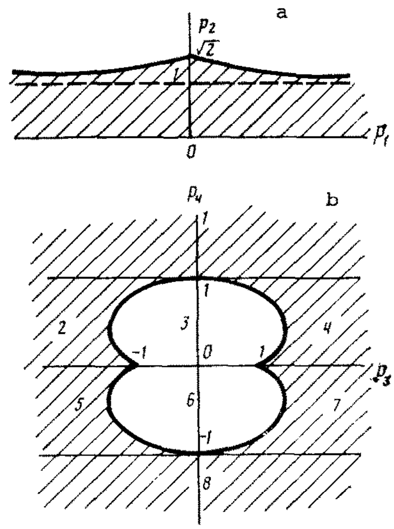

Fig.3

The domain of the parameter space in which the two sets of inequalities are simultaneously satisfied is given by the first system (15) and is shown in Fig.3,a. In a mechanical system whose parameters belong to this domain, all eight types of permanent rotations shown in Fig. 2 will exist.

Let us turn to Eqs.(5). Note that, if u_0 is the root of (5) with $\sigma = 1$, then $-1/u_0$ is the root when $\sigma = -1$, so that we can confine ourselves to the case $\sigma = 1$. By analysing the signs of the function

$$F(u) = (z_2 - z_4)u^4 - 2z_1u^3 - 6z_2u^2 + 2z_1u + (z_2 + z_4) \tag{16}$$

at the points $u = \pm 1, 0$, and as u tends to $\pm\infty$, we can show that Eq.(5) with $\sigma = 1$ always has at least two real roots. For a second pair of real roots to exist, the instant of degeneration to a single double root is critical. This instant is given, apart from the equation $F = 0$, by the condition $F_u' = 0$, i.e., by a system of equations from which u can be eliminated, so that we obtain the dependence

$$p_3 = \pm\sqrt{1 - p_4^{2/3}}\,(1 + 2p_4^{1/3}) \tag{17}$$
$$p_3 = z_1/z_4, \quad p_4 = z_2/z_4$$

connecting the parameters p_3 and p_4. The curve given by (17) (Fig.3,b) bounds the domain of existence of four roots (in the hatched part) and of two roots of Eq.(5) with $\sigma = 1$ and, together with the lines $p_4 = \pm 1$, forms the domains *1 - 8*, in which the function (16) differs in the number of zeros in at least one of the following intervals:

$$u < -1, \quad -1 < u < 0, \quad 0 < u < 1, \quad 1 < u \tag{18}$$

The intervals (18) of u corresponds to the intervals of θ:

$$-\pi < \theta < -\frac{\pi}{2}, \quad -\frac{\pi}{2} < \theta < 0, \quad 0 < \theta < \frac{\pi}{2}, \quad \frac{\pi}{2} < \theta < \pi \tag{19}$$

The zeros u_0 of the function F in the different intervals (18), along with α_0, define the different types of motion. Thus, by analysing the qualitative shape of the graphs of (16) in the domains *1 - 8*, we can at once determine the types of stationary motions that can exist in a mechanical system with the relevant parameters.

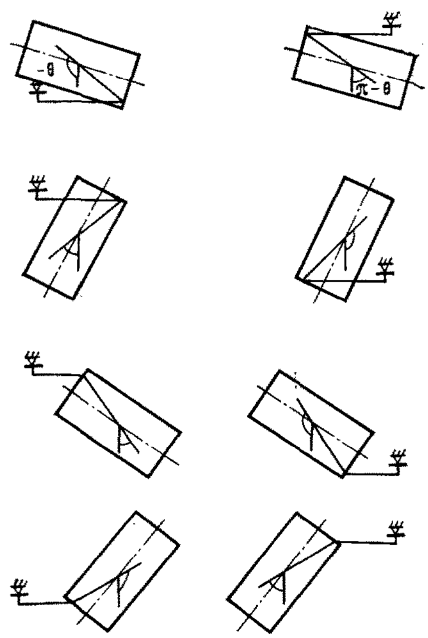

Fig.4

All possible types of permanent rotations with $\alpha_0 = \pi/2$ $(\alpha_0 = -\pi/2)$ are shown on the left (right) of Fig.4. The types of motion on the same horizontal level in Fig.4 correspond to the solutions u_0 and $-1/u_0$ $(|\theta_0(\alpha_0 = \pi/2) - \theta_0(\alpha_0 = -\pi/2)| = \pi)$ of Eqs.(5) with $\sigma = 1$ and $\sigma = -1$. Analysis of the graphs shows that, for a mechanical system with the parameter values of domains *1* and *8*, there is one root in each of intervals (19), and the limiting modes will belong to different types shown in Fig.4. In domains *2, 4, 5, 7*, the total number of limiting modes given by Eqs.(5) will again be eight, though two pairs of them will be of the same type.

To sum up, we have shown that, if the physical parameters of the system are such that the point $(p_1, \ldots, p_4)$ lies in the domain of four-dimensional space which is the Cartesian product of the hatched domains of Fig.3,a, b, we can guarantee the existence of sixteen stationary motions with large ω. If the parameters are such that $|p_4| > 1$, and the point (p_1, p_2) belongs to the hatched domain of Fig.3,a, then the 16 limiting stationary modes will be of different types.

We can use these results to find the number and types of stationary motions at large ω for any mechanical system. But the fact that the parameters are related by

$$p_1 p_4 = p_3 - p_2 \tag{20}$$

means that there is not a set of parameters that defines an actual system corresponding to any point $(p_1, \ldots, p_4)$ of the four-dimensional domain; the set must belong to the hypersurface (20). This means that we cannot easily solve the converse problem, i.e., find the parameters of the system having desired limiting types of terminal modes. An attempt to transform from four- to three-dimensional domains so that a system of three (difficult) inequalities can be solved leads to failure.

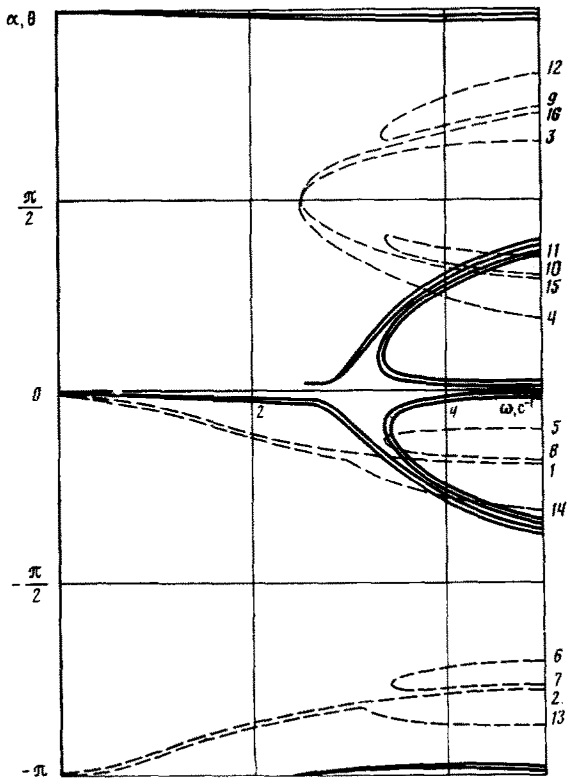

Fig.5

It is probably the case that we can often work with subsets of the domains of Fig.3,a, b, in which some parts can be discarded so as to obtain simple expressions for the boundaries. If we can transform to a subset of at any rate one of the two planes of parameters, i.e., replace the domain of Fig.3,a by a strip, or the domain of Fig.3, b by a rectangle (or augment it up to a rectangle), then the transition from four to three independent parameters takes place automatically.

Let us quote an example of an actual system which, for large ω can correspond to 16 different stationary modes of the permanent rotation type. We suspend on a rod of length $l = 1$ m a uniform disc of radius 45 cm. Let $\delta = 0.77$ rad and $a = 2$ cm; then the disc thickness must be $2a \cos \delta = 2.9$ cm (the rod is clamped to the disc on its front surface). For this system, the dimensionless parameters have the values

$$p_1 = 0.06, \quad p_2 = 0.02, \quad p_3 = -0.06, \quad p_4 = -1.26$$

and thus belong to the hatched domain of Fig.3,a and domain 8 of Fig.3,b. The system must therefore in fact have 16 stationary modes; it can be seen from Fig.5, where we show the curves of α (the continuous curves) and curves of θ (the broken curves) against the angular velocity ω that these modes exist, starting from $\omega \approx 3.4$ sec^{-1} (the numbering of the $\theta(\omega)$ curves corresponds to the numbering of the $\alpha(\omega)$ curves; the numbering is made along the right-hand edge of Fig.5, from top to bottom).

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